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Analysis of thick sandwich construction by a {3, 2}-order theory

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Abstract

This study is an extension of the {1, 2}-order plate theory to a higher order {3, 2} theory. Based on the equivalent single-layer assumptions, the in-plane and transverse displacement components are expressed as cubic and quadratic expansions through the thickness of the sandwich construction. Also, the transverse stress component is assumed to vary as a cubic function through the thickness. Utilizing Reissner's definitions for kinematics of thick plates, the displacement components at any point on the plate are approximated in terms of weighted-average quantities (displacements and rotations) that are functions of the in-plane coordinates. The undetermined coefficients defining the in-plane and transverse displacement fields are then expressed in terms of the weighted-average displacements and rotations and their derivatives by directly employing Reissner's definitions and enforcing the zero transverse-shear-stress conditions on the upper and lower surfaces of the sandwich panel. The coefficients defining the transverse stress component are obtained by requiring the transverse strain component, which is expressed in terms of the unknown coefficients of the transverse stress component from a mixed constitutive relation, to be the least-squares equivalent of the kinematic definition of the transverse strain component. The resulting expressions for the unknown coefficients of the transverse stress component are related to resultant strains and curvatures defined from kinematic relations. The equations of equilibrium and boundary conditions of the sandwich plate based on the {3, 2}-higher-order theory are derived by employing the principles of virtual displacements. The robustness and accuracy of this {3, 2}-order plate theory are established through comparisons with exact solutions available in the literature. The finite element implementation of the present {3, 2}-order plate theory is also discussed. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

A sandwich construction provides high stiffness and high strength-to-weight ratios. It is typically composed of a single soft core with relatively stiff face sheets. The upper and lower face sheets interact

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through the core, transmitting transverse normal and shear stresses. The deformation characteristics of the core and the face sheets are different in nature because of the large differences in stiffness properties. Transverse shear deformations arise from bending, especially when the core is thick and has a relatively low stiffness. Face sheets undergoing unequal displacements can cause the core to experience transverse compression. Transverse compression may also arise due to a locally distributed load on the face sheets.

The first departure from classical plate theory was introduced by Reissner (1944, 1945), who derived his first-order plate theory for homogeneous isotropic plates in equilibrium using an assumed-stress approach. Mindlin (1951) extended this theory to the elastodynamic analysis of plates using a displacement-based first-order theory that also includes rotary inertia effects. Both of these theories take into account the transverse shear deformations in some weighted-average sense.

Since the pioneering works of Reissner (1944, 1945) and Mindlin (1951), numerous first- and higher-order plate theories have been proposed. In general, these theories are based on either equivalent single-layer or discrete-layer assumptions utilizing displacement-based, stress-based, and mixed formulations. In single-layer theory, the displacement components represent the weighted average through the thickness of the sandwich panel. Although the layerwise (discrete-layer) theories (Reddy, 1989; Babuska et al., 1992 and references therein) are more representative of sandwich construction than are single-layer theories, they suffer from an excessive number of field variables in proportion to the number of layers.

Several higher-order shear deformable theories that assume cubic in-plane displacement components and a constant distribution for the transverse displacement component have been developed (Librescu et al., 1987 and references therein). In these theories, the transverse normal stresses are taken into account although no expansion in the transverse direction exists due to the uniform distribution of the transverse displacement component. The effects of normal straining, in addition to the transverse shear deformations, were included by Lo et al. (1977) by introducing a $\{3,2\}$-order displacement theory for homogeneous and laminated plates. Later, Reddy (1990) established a correlation between several versions of the $\{3,2\}$-order theory. A major shortcoming of these theories arises from the inclusion of a large number of plate displacement variables and the complexity of natural boundary conditions.

Tessler (1993) developed a $\{1,2\}$-order single-layer theory that eliminates the above-mentioned shortcomings, leading to the determination of complete stress and strain fields with only a few displacement variables. This theory assumes linear expansion of in-plane displacements and a special parabolic form for the transverse displacement component in the thickness direction. Similarly, Cook and Tessler (1998) formulated a $\{3,2\}$ higher-order theory for sandwich beams. Their formulation assumes cubic expansion for the in-plane displacements; hence, the correct variation of the in-plane stresses and the transverse shear stresses (from the equilibrium equations) is captured. In both of these formulations, a cubic variation of the transverse stress field is assumed, thus satisfying the continuity of an interlaminar transverse stress field, as well as the transverse stress equilibrium equations on the plate's upper and lower surfaces.

Based on the investigations presented by Tessler (1993) and Cook and Tessler (1998), the present formulation extends their work to the analysis of sandwich panels by a $\{3,2\}$-order plate theory, which is analogous to the $\{3,2\}$-order beam theory of Cook and Tessler. The development of the $\{3,2\}$-order plate theory and the derivation of the equations of equilibrium and boundary conditions are presented in the following sections. The present theory is verified using Pagano's (1970) exact solutions obtained for simply supported and square sandwich construction subjected to double sinusoidal loading. Sandwich constructions with carbon/epoxy face sheets and a core of PVC or a honeycomb material are considered. The results show remarkable agreement with the exact solution provided by Pagano for thick sandwich panels with length-to-thickness ratios equal to four. The applicability of the present formulation to finite element methods is discussed in the conclusions.

2. Formulation

The geometric and loading descriptions of a thick sandwich panel are illustrated in Fig. 1. The panel has an arbitrary planar geometry and a thickness $2h$. Along the vertical edges, where the vector $\mathbf{n}$ is used to indicate the unit normal of the edge, the sandwich panel is under the action of arbitrary traction forces T_x, T_y, and T_z in the x-, y-, and z-directions, respectively. The upper and lower faces of the panel are subjected to normal stresses, $q^+(x,y)$ and $q^-(x,y)$, as shown in Fig. 1. The displacements at any point on the panel are represented by in-plane displacement components, $u_x(x,y,z)$ and $u_y(x,y,z)$, and the transverse displacement component, $u_z(x,y,z)$. The $\{3,2\}$-order plate theory implies that the in-plane displacements vary cubically and the transverse displacement component quadratically across the thickness (z-direction) of the panel. Thus, the displacement components of the sandwich panel are defined in the form

$$u_x(x,y,z) = u_0(x,y) + u_1(x,y)\xi + u_2(x,y)\xi^2 + u_3(x,y)\xi^3 \tag{1a}$$

$$u_y(x,y,z) = v_0(x,y) + v_1(x,y)\xi + v_2(x,y)\xi^2 + v_3(x,y)\xi^3 \tag{1b}$$

$$u_z(x,y,z) = w_0(x,y) + w_1(x,y)\xi + w_2(x,y)\left(\xi^2 - \tfrac{1}{5}\right) \tag{1c}$$

where $\xi = z/h$. The constant $-1/5$ in the expression for u_z appears as a result of Reissner's (1945) definition for the weighted average of the transverse displacement component, u_z.

As proposed by Tessler (1993) the weighted average quantities through the thickness of the panel are defined as

$$(u(x,y), v(x,y)) = \frac{1}{2h}\int_{-h}^{h}(u_x(x,y,z), u_y(x,y,z))\,\mathrm{d}z \tag{2a}$$

$$(\theta_x(x,y), \theta_y(x,y)) = \frac{3}{2h^3}\int_{-h}^{h}(u_y(x,y,z), u_x(x,y,z))\,\mathrm{d}z \tag{2b}$$

$$w(x,y) = \frac{3}{4h}\int_{-h}^{h}u_z(x,y,z)(1-\xi^2)\,\mathrm{d}z \tag{2c}$$

Enforcing the definition for weighted-average kinematic variables results in the expressions for u_i and v_i $(i = 2,3)$ in terms of the weighted-average kinematic variables, u, v, θ_x, and θ_y, and the unknown

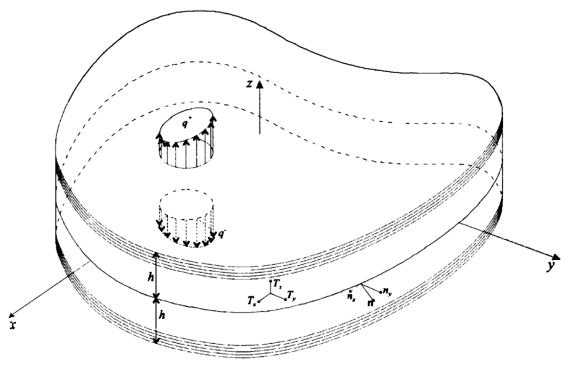

Fig. 1. Geometric description of a flat sandwich panel subjected to arbitrary edge loading and normal stress at the faces.

coefficients, u_i and v_i $(i = 0, 1)$. The remaining unknown coefficients, u_i and v_i $(i = 0, 1)$ are determined by imposing zero shear traction conditions on the upper and lower surfaces of the sandwich panel,

$$\sigma_{xz}(x, y, \mp h) = \sigma_{yz}(x, y, \mp h) = 0 \tag{3}$$

Applying these conditions through the constitutive relations leads to the requirement that the shear strain components vanish,

$$\gamma_{xz}(x, y, \mp h) = \gamma_{yz}(x, y, \mp h) = 0 \tag{4}$$

Enforcing these conditions results in the determination of the unknown functions, u_i and v_i $(i = 0, 1)$.

After determining the coefficients u_i and v_i $(i = 0, 1, 2, 3)$, the displacements, u_x and u_y, are expressed in terms of the average quantities $(u, v, \theta_x, \theta_y)$ and (w_0, w_1, w_2) as

$$u_x(x, y, z) = P_0(\xi)u(x, y) + P_1(\xi)\theta_y(x, y) + P_2(\xi)hw_{1,x}(x, y) + P_3(\xi)\left[\tfrac{5}{4}\left(\theta_y(x, y) + w_{0,x}(x, y)\right) + w_{2,x}(x, y)\right] \tag{5a}$$

$$u_y(x, y, z) = P_0(\xi)v(x, y) + P_1(\xi)\theta_x(x, y) + P_2(\xi)hw_{1,y}(x, y) + P_3(\xi)\left[\tfrac{5}{4}\left(\theta_x(x, y) + w_{0,y}(x, y)\right) + w_{2,y}(x, y)\right] \tag{5b}$$

in which

$$P_0 = 1, \quad P_1 = h\xi, \quad P_2 = \left(\frac{1}{6} - \frac{\xi^2}{2}\right), \quad P_3 = h\left(\frac{\xi}{5} - \frac{\xi^3}{3}\right) \tag{6}$$

differ from the Legendre polynomials only by scaling factors.

2.1. Kinematic representation of strain components

The strain components, expressed in the form

$$\varepsilon_{xx} = u_{x,x}, \quad \varepsilon_{yy} = u_{y,y}, \quad \varepsilon_{zz} = u_{z,z} \tag{7a}$$

$$\gamma_{yz} = u_{y,z} + u_{z,y}, \quad \gamma_{xz} = u_{x,z} + u_{z,x}, \quad \gamma_{xy} = u_{x,y} + u_{y,x} \tag{7b}$$

can be rewritten as

$$\varepsilon_{xx} = P_0\varepsilon_{xx0} + P_1\kappa_{xx0} + P_2\varepsilon_{xx1} + P_3\kappa_{xx1} \tag{8a}$$

$$\varepsilon_{yy} = P_0\varepsilon_{yy0} + P_1\kappa_{yy0} + P_2\varepsilon_{yy1} + P_3\kappa_{yy1} \tag{8b}$$

$$\gamma_{xy} = P_0\gamma_{xy0} + P_1\kappa_{xy0} + P_2\gamma_{xy1} + P_3\kappa_{xy1} \tag{8c}$$

$$\varepsilon_{zz} = \varepsilon_{zz0}P_0 + 2\kappa_{zz0}P_1, \qquad (\gamma_{yz}, \gamma_{xz}) = \tfrac{5}{4}(1 - \xi^2)(\gamma_{yz0}, \gamma_{xz0}) \tag{8d}$$

in which the resultant strains and curvatures are defined as

$$\varepsilon_{xx0} = u_{,x}, \quad \varepsilon_{yy0} = v_{,y}, \quad \varepsilon_{zz0} = w_1/h, \quad \gamma_{xy0} = u_{,y} + v_{,x} \tag{9a}$$

$$\varepsilon_{xx1} = hw_{1,xx}, \quad \varepsilon_{yy1} = hw_{2,xx}, \quad \gamma_{xy1} = 2hw_{1,xy} \tag{9b}$$

$$\kappa_{xx0} = \theta_{y,x}, \quad \kappa_{yy0} = \theta_{x,y}, \quad \kappa_{zz0} = w_2/h^2, \quad \kappa_{xy0} = \theta_{x,x} + \theta_{y,y} \tag{9c}$$

$$\kappa_{xx1} = \tfrac{5}{4}(\theta_{y,x} + w_{0,xx}) + w_{2,xx} \tag{9d}$$

$$\kappa_{yy1} = \tfrac{5}{4}(\theta_{y,x} + w_{0,yy}) + w_{2,yy} \tag{9e}$$

$$\kappa_{xy1} = \tfrac{5}{4}(\theta_{x,x} + \theta_{y,y} + 2w_{0,xy}) + 2w_{2,xy} \tag{9f}$$

$$\gamma_{yz0} = w_{0,y} + \theta_x, \qquad \gamma_{xz} = w_{0,x} + \theta_y \tag{9g}$$

2.2. Assumed transverse stress component

Although the transverse strain component, ε_{zz}, is expressed based on the kinematic relation in terms of average quantities, it leads to a discontinuous interlaminar transverse normal stress field, σ_{zz}, when obtained through constitutive relations. This undesirable condition also appears for the transverse shear stresses, σ_{yz} and σ_{xz}; however, they can be recovered with reasonable accuracy from the stress equilibrium equations as

$$\sigma_{xz,z} = -\sigma_{xx,x} - \sigma_{xy,y} \tag{10a}$$

$$\sigma_{yz,z} = -\sigma_{yy,y} - \sigma_{xy,x} \tag{10b}$$

provided that the derivatives of the in-plane stresses are obtained with high accuracy.

In order to achieve a continuous through-thickness distribution of the transverse normal stress, a cubic polynomial is assumed for σ_{zz} as

$$\sigma_{zz}(x,y,z) = \sigma_{zz0}(x,y) + \phi\sigma_{zz1}(x,y) \tag{11}$$

where $\phi = \xi - \xi^3/3$ and σ_{zz0} and σ_{zz1} are unknown functions of x and y. This assumption was proposed by Tessler (1993) and applied successfully by Cook and Tessler (1998) for sandwich beams. Note that σ_{zz} in Eq. (11) satisfies the exact stress equilibrium equation

$$\sigma_{zz,z} = -\sigma_{xz,x} - \sigma_{yz,y} \tag{12}$$

for the case of zero transverse shear stresses on the upper and lower surfaces, leading to

$$\sigma_{zz,z}(x,y,\pm h) = 0 \tag{13}$$

This condition states that the slope of the transverse stress σ_{zz} at the upper and lower faces of the panel is zero. Furthermore, three-dimensional elasticity solutions for σ_{zz} due to mechanical pressure loads closely resemble a cubic distribution in the thickness direction.

In order to utilize Eq. (11) in the stress–strain relations, a mixed form of constitutive equations is employed as

$$\begin{Bmatrix} \sigma_{xx}^{(k)} \\ \sigma_{yy}^{(k)} \\ \varepsilon_{zz}^{(k)} \\ \sigma_{yz}^{(k)} \\ \sigma_{xz}^{(k)} \\ \sigma_{xy}^{(k)} \end{Bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & R_{13} & 0 & 0 & \bar{C}_{16} \\ \bar{C}_{12} & \bar{C}_{22} & R_{23} & 0 & 0 & \bar{C}_{26} \\ -R_{13} & -R_{23} & S_{33} & 0 & 0 & -R_{63} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ \bar{C}_{16} & \bar{C}_{26} & R_{63} & 0 & 0 & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \sigma_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{14}$$

where the superscript (k) indicates that the quantities are obtained from the stress–strain relation in the kth layer, and strain and stress quantities without the superscript on the right-hand side of Eq. (14) are directly expressed in terms of strain resultants. The components of the material property matrix are defined as

$$\bar{C}_{ij}^{(k)} = C_{ij}^{(k)} - R_{i3}^{(k)}C_{j3}^{(k)} \quad (i,j = 1,2,6) \tag{15a}$$

$$R_{i3}^{(k)} = C_{i3}^{(k)} S_{33}^{(k)} \quad \text{with } S_{33}^{(k)} = \frac{1}{C_{33}^{(k)}} \quad (i = 1, 2, 6) \tag{15b}$$

The transverse strain in the kth layer can then be expressed as

$$\varepsilon_{zz}^{(k)} = -R_{13}^{(k)} \varepsilon_{xx} - R_{23}^{(k)} \varepsilon_{yy} - R_{63}^{(k)} \gamma_{xy} + S_{33}^{(k)} \sigma_{zz} \tag{16}$$

The unknown functions of the transverse stress component, σ_{zz0} andσ_{zz1}, are determined such that the transverse strain component $\varepsilon_{zz}^{(k)}$ becomes the least-squares equivalent of the kinematic definition of the transverse strain component (i.e., $\varepsilon_{zz} = u_{z,z}$). This is accomplished by minimizing the total error between $\varepsilon_{zz}^{(k)}$ and ε_{zz} through the thickness of the panel. The error function is defined as

$$\min \left[\int_{-h}^{h} (\varepsilon_{zz}^{(k)} - \varepsilon_{zz})^2 \, \mathrm{d}z \right] \tag{17}$$

Minimization of Eq. (17) with respect to σ_{zz0} and σ_{zz1} results in the expressions for σ_{zz0} and σ_{zz1} in the form

$$\left\{ \begin{array}{c} \sigma_{zz0} \\ \sigma_{zz1} \end{array} \right\} = \begin{bmatrix} \mathbf{q}'_{\varepsilon} & \mathbf{q}''_{\kappa} \\ \mathbf{q}''_{\varepsilon} & \mathbf{q}''_{\kappa} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{\epsilon} \\ \boldsymbol{\kappa} \end{array} \right\} \tag{18}$$

in which

$$\mathbf{q}'_{a} = \left\{ q'_{1\alpha} \quad q'_{2\alpha} \quad \cdots \quad q'_{7\alpha} \right\}, \quad \mathbf{q}''_{\alpha} = \left\{ q''_{1\alpha} \quad q''_{2\alpha} \quad \cdots \quad q''_{7\alpha} \right\}, \quad (\alpha = \varepsilon, \kappa) \tag{19}$$

and

$$\boldsymbol{\epsilon}^{\mathrm{T}} = \left\{ \varepsilon_{xx0}, \varepsilon_{yy0}, \varepsilon_{zz0}, \gamma_{xy0}, \varepsilon_{xx1}, \varepsilon_{yy1}, \gamma_{xy1} \right\} \tag{20a}$$

$$\boldsymbol{\kappa}^{\mathrm{T}} = \left\{ \kappa_{xx0}, \kappa_{yy0}, \kappa_{zz0}, \kappa_{xy0}, \kappa_{xx1}, \kappa_{yy1}, \kappa_{xy1} \right\} \tag{20b}$$

The explicit expressions for $\mathbf{q}'_{\alpha}$ and $\mathbf{q}''_{\alpha} (\alpha = \varepsilon, \kappa)$ are given in Appendix A.

2.3. Equilibrium equations

The equilibrium equations for the plate are derived based on the principle of virtual work,

$$\delta U = \delta W_{\mathrm{E}} \tag{21}$$

in which δU represents the work done by the internal forces over virtual (arbitrary) displacements in the material and δW_{E} is the work done by the external forces over virtual displacements on the boundaries. For a sandwich panel subjected to traction forces as shown in Fig. 1, application of the principle of virtual work leads to

$$\begin{aligned} &\int_A \int_{-h}^{h} \left(\sigma_{xx}^{(k)} \delta\varepsilon_{xx} + \sigma_{yy}^{(k)} \delta\varepsilon_{yy} + \sigma_{zz} \delta\varepsilon_{zz}^{(k)} + \sigma_{yz}^{(k)} \delta\gamma_{yz} + \sigma_{xz}^{(k)} \delta\gamma_{xz} + \sigma_{xy}^{(k)} \delta\gamma_{xy} \right) \mathrm{d}z \, \mathrm{d}A \\ &\quad = \int_A (q^+ \delta u_z(x, y, h)) \, \mathrm{d}A - \int_A (q^- \delta u_z(x, y, -h)) \, \mathrm{d}A + \oint_{\ell} \int_{-h}^{h} \left\{ T_x \delta u_x + T_y \delta u_y + T_z \delta u_z \right\} \mathrm{d}z \, \mathrm{d}\ell \end{aligned} \tag{22}$$

where A represents the area of the plate's middle surface. The arc length along the boundary of the panel is denoted by ℓ. Substituting for the displacement and strain components from Eqs. (1c), (5), (8a)–(8d), and (16) and integrating through the thickness in Eq. (22) results in

$$\int_A \{N_{xx0}\delta\varepsilon_{xx0} + N_{yy0}\delta\varepsilon_{yy0} + N_{zz0}\delta\varepsilon_{zz0} + N_{xy0}\delta\gamma_{xy0} + N_{xx1}\delta\varepsilon_{xx1} + N_{yy1}\delta\varepsilon_{yy1} + N_{xy1}\delta\gamma_{xy1} + M_{xx0}\delta\kappa_{xx0}$$
$$+ M_{yy0}\delta\kappa_{yy0} + M_{zz0}\delta\kappa_{zz0} + M_{xy0}\delta\kappa_{xy0} + M_{xx1}\delta\kappa_{xx1} + M_{yy1}\delta\kappa_{yy1} + M_{xy1}\delta\kappa_{xy1} + Q_{yz0}\delta\gamma_{yz0} + Q_{xz0}\delta\gamma_{xz0}\}\mathrm{d}A$$
$$= \int_A \left\{(q^+ - q^-)\delta w_0 + (q^+ + q^-)\delta w_1 + \frac{4}{5}(q^+ - q^-)\delta w_2\right\}\mathrm{d}A + \oint_\ell \left\{\hat{T}_{x0}\delta u + \hat{M}_{x0}\delta\theta_y + \hat{T}_{x1}h\delta w_{1,x}\right.$$
$$+ \hat{M}_{x1}\delta\left[\frac{5}{4}(\theta_y + w_{0,x}) + w_{2,x}\right] + \hat{T}_{y0}\delta v + \hat{M}_{y0}\delta\theta_x + \hat{T}_{y1}h\delta w_{1,y} + \hat{M}_{y1}\delta\left[\frac{5}{4}(\theta_x + w_{0,y}) + w_{2,y}\right]$$
$$\left. + \hat{Q}_{nz0}\delta w_0 + \hat{Q}_{nz1}\delta w_1 + \hat{Q}_{nz2}\delta w_2\right\}\mathrm{d}\ell \tag{23}$$

The expressions for the load vectors $\hat{T}_{\alpha i}$, $\hat{M}_{\alpha i}$, and $\hat{Q}_{nzi}$ $(\alpha = x, y; i = 0, 1, 2)$ are given in Appendix A. The resultant stresses, $N_{\alpha\beta i}$, Q_{yz0}, and Q_{xz0}, and the resultant moments, $M_{\alpha\beta i}$ $(\alpha, \beta = x, y; i = 0, 1)$ are obtained by integrating the product of appropriate stress components with appropriate functions in terms of ξ, resulting in the following constitutive relation:

$$\begin{Bmatrix} \mathbf{N} \\ \mathbf{M} \\ \mathbf{Q} \end{Bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma} \end{Bmatrix} \tag{24}$$

where

$$\mathbf{N}^{\mathrm{T}} = \{N_{xx0}, N_{yy0}, N_{zz0}, N_{xy0}, N_{xx1}, N_{yy1}, N_{xy1}\} \tag{25a}$$

$$M^{\mathrm{T}} = \{M_{xx0}, M_{yy0}, M_{zz0}, M_{xy0}, M_{xx1}, M_{yy1}, M_{xy1}\} \tag{25b}$$

$$\mathbf{Q}^{\mathrm{T}} = \{Q_{yz0}, Q_{xz0}\} \tag{25c}$$

with $\boldsymbol{\epsilon}$ and $\boldsymbol{\kappa}$ defined in Eqs. (20a) and (20b) and

$$\boldsymbol{\gamma}^{\mathrm{T}} = \{\gamma_{yz0}, \gamma_{xz0}\} \tag{26}$$

The matrices **A**, **D**, and **B** are defined as

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & k_{z0}A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ & A_{22} & k_{z0}A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ & & k_{z0}^2 A_{33} & k_{z0}A_{34} & k_{z0}A_{35} & k_{z0}A_{36} & k_{z0}A_{37} \\ & & & A_{44} & A_{45} & A_{46} & A_{47} \\ & & & & A_{55} & A_{56} & A_{57} \\ \text{sym.} & & & & & A_{66} & A_{67} \\ & & & & & & A_{77} \end{bmatrix} \tag{27a}$$

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & k_{z1}D_{13} & D_{14} & D_{15} & D_{16} & D_{17} \\ & D_{22} & k_{z1}D_{23} & D_{24} & D_{25} & D_{26} & D_{27} \\ & & k_{z1}^2 D_{33} & k_{z1}D_{34} & k_{z1}D_{35} & k_{z1}D_{36} & k_{z1}D_{37} \\ & & & D_{44} & D_{45} & D_{46} & D_{47} \\ & & & & D_{55} & D_{56} & D_{57} \\ \text{sym.} & & & & & D_{66} & D_{67} \\ & & & & & & D_{77} \end{bmatrix} \tag{27b}$$

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & k_{z1}B_{13} & B_{14} & B_{15} & B_{16} & B_{17} \\ B_{21} & B_{22} & k_{z1}B_{23} & B_{24} & B_{25} & B_{26} & B_{27} \\ k_{z0}B_{31} & k_{z0}B_{32} & k_{z0}k_{z1}B_{33} & k_{z0}B_{34} & k_{z0}B_{35} & k_{z0}B_{36} & k_{z0}B_{37} \\ B_{41} & B_{42} & k_{z1}B_{43} & B_{44} & B_{45} & B_{46} & B_{47} \\ B_{51} & B_{52} & k_{z1}B_{53} & B_{54} & B_{55} & B_{56} & B_{57} \\ B_{61} & B_{62} & k_{z1}B_{63} & B_{64} & B_{65} & B_{66} & B_{67} \\ B_{71} & B_{72} & k_{z1}B_{73} & B_{74} & B_{75} & B_{76} & B_{77} \end{bmatrix} \tag{27c}$$

and the matrix **G** has the form

$$\mathbf{G} = \begin{bmatrix} k_{yz}^2 G_{11} & k_{yz}k_{xz}G_{12} \\ k_{yz}k_{xz}G_{12} & k_{xz}^2 G_{22} \end{bmatrix} \tag{27d}$$

where k_{z0} and k_{z1} denote the transverse correction factors (Cook and Tessler, 1998) and k_{yz} and k_{xz} are known as the shear correction factors. The shear correction factors improve the global transverse displacements of the plate. The role of k_{z0} and k_{z1} is to improve the transverse stretching of the plate. These correction factors are obtained using the approach described by Cook (1997). The expressions for A_{ij}, B_{ij}, D_{ij} $(i,j = 1,7)$, and G_{ij} $(i,j = 1,2)$ are summarized in Appendix A.

Substituting for the resultant strains and curvatures from Eqs. (9a)–(9g) and integrating the area integrals of Eq. (12) by parts and applying the Gauss theorem to appropriate terms result in the Euler equations of equilibrium and boundary conditions for the sandwich panel:

On $\alpha \in A$ $(\alpha = x, y)$,

$$N_{xx0,x} + N_{xy0,y} = 0 \tag{28a}$$

$$N_{yy0,y} + N_{xy0,x} = 0 \tag{28b}$$

$$\tfrac{5}{4}(M_{xx1,xx} + M_{yy1,yy} + 2M_{xy1,xy}) - Q_{xz0,x} - Q_{yz0,y} - \bar{q}_1 = 0 \tag{28c}$$

$$\tfrac{5}{4}(M_{yy1,y} + M_{xy1,x}) + M_{yy0,y} + M_{xy0,x} - Q_{yz0} = 0 \tag{28d}$$

$$\tfrac{5}{4}(M_{xx1,x} + M_{xy1,y}) + M_{xx0,x} + M_{xy0,y} - Q_{xz0} = 0 \tag{28e}$$

$$\frac{N_{zz0}}{h} + h(N_{xx1,xx} + N_{yy1,yy} + 2N_{xy1,xy}) - \bar{q}_2 = 0 \tag{28f}$$

$$\frac{M_{zz0}}{h^2} + (M_{xx1,xx} + M_{yy1,yy} + 2M_{xy1,xy}) - \tfrac{4}{5}\bar{q}_1 = 0 \tag{28g}$$

where

$$\bar{q}_1 = q^+ - q^-, \qquad \bar{q}_2 = q^+ + q^- \tag{28h}$$

On $\alpha \in \ell$ $(\alpha = x, y)$,

$$N_{xx0}n_x + N_{xy0}n_y = \hat{T}_{x0} \quad \text{or} \quad \delta u = 0 \tag{29a}$$

$$N_{yy0}n_y + N_{xy0}n_x = \hat{T}_{y0} \quad \text{or} \quad \delta v = 0 \tag{29b}$$

$$Q_{xz0}n_x + Q_{yz0}n_y - \tfrac{5}{4}(M_{xx1,x}n_x + M_{xy1,x}n_y) - \tfrac{5}{4}(M_{yy1,y}n_y + M_{xy1,y}n_x) = \hat{Q}_{nz0} \quad \text{or} \quad \delta w_0 = 0 \tag{29c}$$

$$M_{yy0}n_y + M_{xy0}n_x = \hat{M}_{y0} \quad \text{or} \quad \delta\theta_x = 0 \tag{29d}$$

$$M_{xx0}n_x + M_{xy0}n_y = \hat{M}_{x0} \quad \text{or} \quad \delta\theta_y = 0 \tag{29e}$$

$$h(N_{xx1,x}n_x + N_{yy1,y}n_y + N_{xy1,x}n_y + N_{xy1,y}n_x) = -\hat{Q}_{nz1} \quad \text{or} \quad \delta w_1 = 0 \tag{29f}$$

$$N_{xx1}n_x + N_{xy1}n_y = \hat{T}_{x1} \quad \text{or} \quad \delta w_{1,x} = 0 \tag{29g}$$

$$N_{yy1}n_y + N_{xy1}n_x = \hat{T}_{y1} \quad \text{or} \quad \delta w_{1,y} = 0 \tag{29h}$$

$$M_{xx1,x}n_x + M_{xy1,x}n_y + M_{yy1,y}n_y + M_{xy1,y}n_x = -\hat{Q}_{nz2} \quad \text{or} \quad \delta w_2 = 0 \tag{29i}$$

$$M_{xx1}n_x + M_{xy1}n_y = \hat{M}_{x1} \quad \text{or} \quad \delta\left[\tfrac{5}{4}(\theta_y + w_{0,x}) + w_{2,x}\right] = 0 \tag{29j}$$

$$M_{yy1}n_y + M_{xy1}n_x = \hat{M}_{y1} \quad \text{or} \quad \delta\left[\tfrac{5}{4}(\theta_x + w_{0,y}) + w_{2,y}\right] = 0 \tag{29k}$$

3. Simply supported sandwich panel

In order to validate the present plate theory analytically, a simply supported sandwich panel is considered (Fig. 2). The panel has a square, planar geometry described by $a = b = 4h$, with $h = 0.5$ in. The face sheets at the bottom and top of the core are of equal thickness, $h_f = 0.2h$; hence, the core thickness, h_c, is 80% of the total panel thickness. The panel is subjected to double-sinusoidal external pressure on the top surface, with magnitude p_0, as depicted in Fig. 2.

The face sheet material is composed of carbon/epoxy plies, with stacking sequence $[0/90]_5$ (the bottom-most layer and the layer on the top have a ply angle of 0° with respect to the x-axis). Each ply has Young's moduli of $E_L = 22.9 \times 10^6$ psi and $E_T = 1.39 \times 10^6$ psi, shear moduli of $G_{LT} = 0.864 \times 10^6$ psi and $G_{TT} = 0.368 \times 10^6$ psi, and Poisson's ratios of $\nu_{LT} = 0.32$ and $\nu_{TT} = 0.48$. In order to investigate the effect of core-face sheet interaction, two different materials are considered for the core. The first one is made of a soft PVC material with $E_c = 15.03 \times 10^3$ psi and $\nu_c = 0.3$, with the face sheet being about 10^3 times stiffer than the core in both shear and in-plane stresses. The second core material is made of a titanium honeycomb material with $E_1 = 62.36$ psi, $E_2 = 41.27$ psi, $E_3 = 345 \times 10^3$ psi, $G_{12} = 1140$ psi, $G_{23} = 56.7 \times 10^3$ psi,

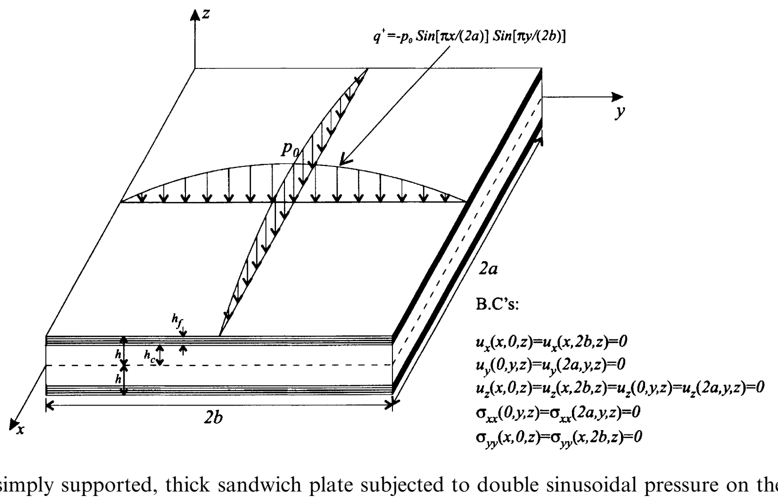

Fig. 2. A simply supported, thick sandwich plate subjected to double sinusoidal pressure on the top face.

$G_{13} = 75.1 \times 10^3$ psi, $\nu_{12} = 1.23$, $\nu_{23} = 5.6 \times 10^{-5}$, and $\nu_{13} = 3.7 \times 10^{-5}$. The honeycomb has a higher transverse stiffness but lower in-plane stiffnesses than PVC.

For this problem, the equilibrium equations, Eqs. (28a)–(28h), can be integrated by assuming the following displacements and rotations:

$$u(x,y) = U\cos\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi y}{2b}\right) \tag{30a}$$

$$v(x,y) = V\sin\left(\frac{\pi x}{2a}\right)\cos\left(\frac{\pi y}{2b}\right) \tag{30b}$$

$$[w_0(x,y), w_1(x,y), w_2(x,y)] = [W_0, W_1, W_2]\sin\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi y}{2b}\right) \tag{30c–e}$$

$$\theta_x(x,y) = \Phi_x\sin\left(\frac{\pi x}{2a}\right)\cos\left(\frac{\pi y}{2b}\right) \tag{30f}$$

$$\theta_y(x,y) = \Phi_y\cos\left(\frac{\pi x}{2a}\right)\sin\left(\frac{\pi y}{2b}\right) \tag{30g}$$

where U, V, W_0, W_1, W_2, Φ_x, and Φ_y are unknown amplitudes of the displacements and rotations. Note that Eqs. (30a)–(30g) satisfy both displacement and force boundary conditions, as given by Eqs. (29a)–(29k). In order to determine the unknown amplitudes, equilibrium equations, Eqs. (28a)–(28h), are expressed in terms of displacement components through substitutions of Eqs. (9a)–(9g) into Eq. (24) and Eq. (24) into Eqs. (28a)–(28h), resulting in the equations of equilibrium in terms of displacements. Eqs. (30a)–(30g) are incorporated into these displacement equilibrium equations to obtain solutions for the unknown amplitudes. Note that the applicability of the solution form given in Eqs. (30a)–(30g) is limited to sandwich panels with cross-ply lamination configurations. The resulting equations will yield a system of seven equations for seven unknown amplitudes, with the right-hand side of the equations containing the amplitude of the applied load.

The results of the present theory are compared against the exact solutions provided by Pagano (1970) for bi-directional simply supported sandwich plates. Three critical locations on the plate are selected to compare the two solutions. The in-plane strains and stresses and the transverse strains, stresses, and displacements are examined at the center ($x = a$, $y = b$) of the plate. The transverse shear strain and stress components, γ_{xz} and σ_{xz}, and the in-plane displacement component, u_x, are evaluated at $x = 2a$ and $y = b$. Similarly, the transverse shear strain and stress components, γ_{yz} and σ_{yz}, and the in-plane displacement component, u_y, are evaluated at $x = a$ and $y = 2b$.

Fig. 3 shows a comparison of in-plane and transverse stresses at the center and the transverse shear stresses at $x = 2a$, $y = b$ and $x = a$, $y = 2b$ for a sandwich panel with a PVC core. Also shown in this figure is a comparison of transverse displacement components at the center of the plate. As shown in Fig. 3a and b, the results from both the present analysis and the exact solution by Pagano (1970) are in excellent agreement for in-plane stresses, σ_{xx} and σ_{yy} (note that $\sigma_{xy} = 0$ everywhere due to the orthogonal symmetry of both the geometry and layup of the materials). As mentioned in Section 2.2, the transverse shear stresses are computed by integrating Eqs. (10a) and (10b) through the thickness. The accuracy achieved for the in-plane stresses gives rise to remarkable agreement of the transverse shear stress computations between the present analysis and the exact solution. As seen in Figs. 3c and 3d, the present $\{3,2\}$ higher-order plate theory captures the correct variation of transverse shear stresses both along the core and along the face sheets.

Comparisons of plots for the transverse stress component, σ_{zz}, and transverse displacement, w_c, at the center of the panel are illustrated in Fig. 3e and f, respectively. Favorable agreement between the two

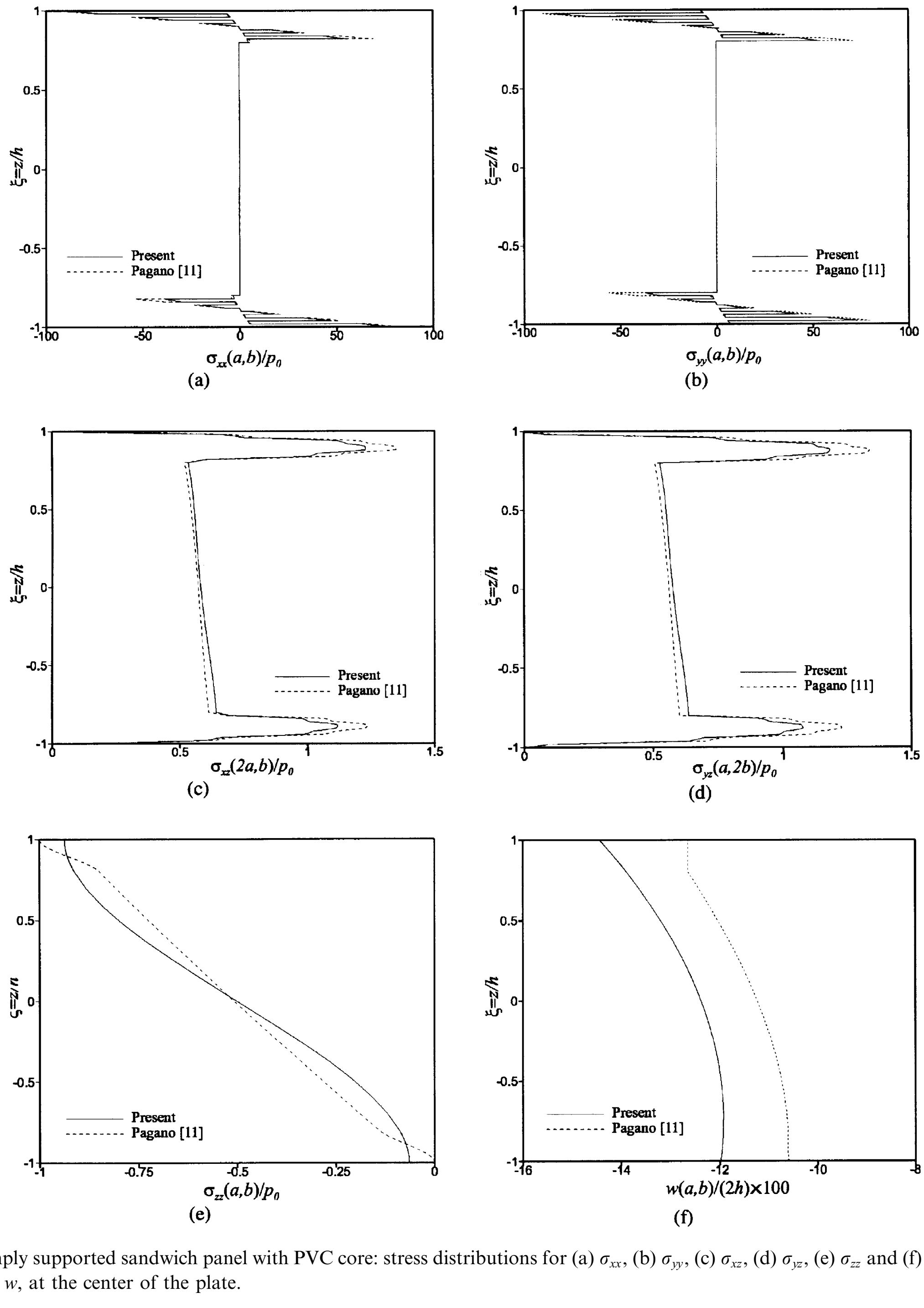

Fig. 3. Simply supported sandwich panel with PVC core: stress distributions for (a) σ_{xx}, (b) σ_{yy}, (c) σ_{xz}, (d) σ_{yz}, (e) σ_{zz} and (f) the vertical deflection, w, at the center of the plate.

solutions is obtained for a plate with $a/h = 4$ via the computation of appropriate transverse correction factors, as described by Cook (1997).

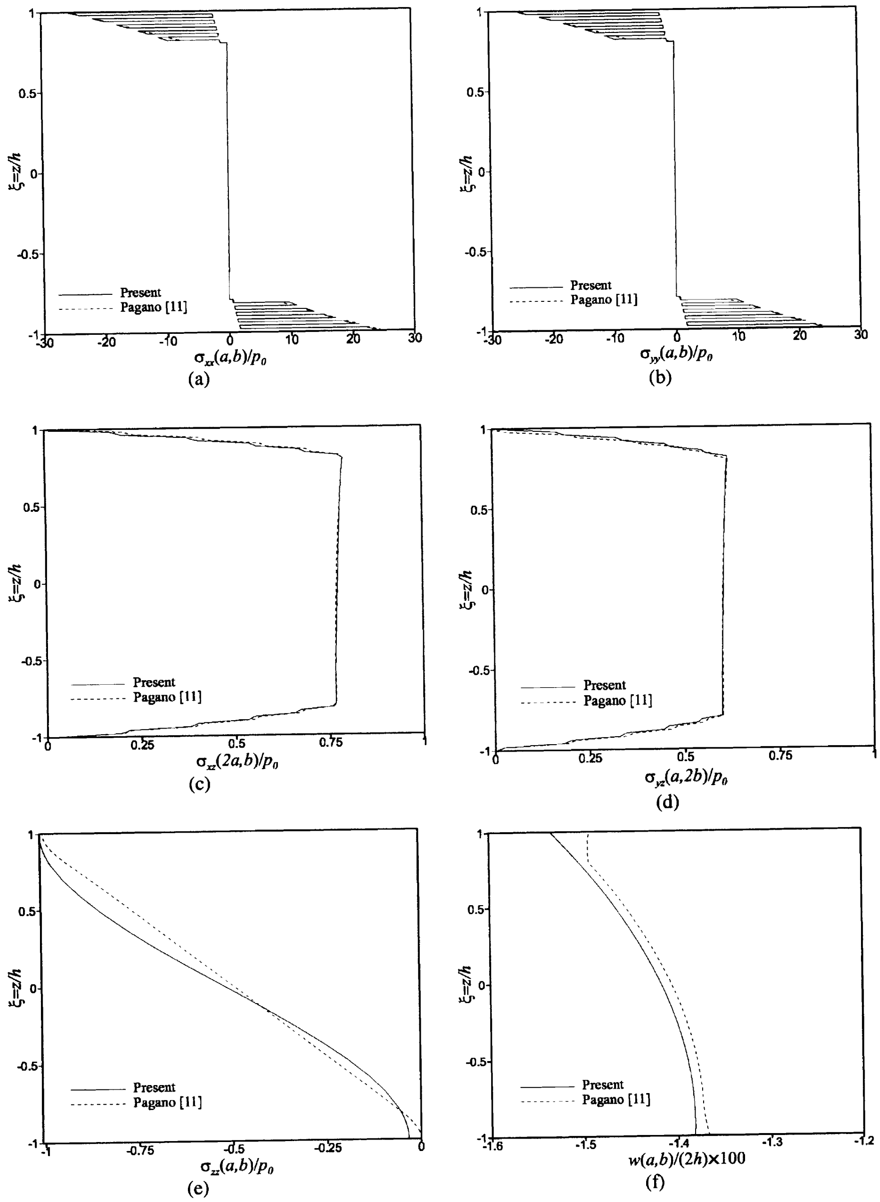

Fig. 4. Simply supported sandwich panel with honeycomb core: stress distributions for (a) σ_{xx}, (b) σ_{yy}, (c) σ_{xz}, (d) σ_{yz}, (e) σ_{zz} and (f) the vertical deflection, w, at the center of the plate.

In the case of a sandwich panel with a honeycomb core (Fig. 4), it is observed that the present $\{3, 2\}$ higher-order theory yields almost the same accuracy as in the case of a sandwich panel with a PVC core. The difference is observed in the responses of these two material systems. Unlike the sandwich panel with the PVC core, the honeycomb-core panel can take high transverse shear stresses (Fig. 4c and d), thus causing tension in the lower face sheet and compression in the upper face sheet. In the case of the PVC core, however, the low transverse shear stiffness of the core causes the face sheets to bend rather than to stretch or contract (as observed in the case of a honeycomb core).

4. Remarks on finite element implementation

Although the present $\{3,2\}$-order plate theory appears to be robust for analyzing sandwich as well as thin-to-thick regime laminated composite plates, the applicability of the present theory to the finite element method presents difficulties in terms of the choice of shape functions. As noticed in the derivation of the equilibrium equations, the boundary conditions given in Eq. (29) suggest that the derivatives of the element interpolation functions for transverse displacement fields, w_0, w_1, and w_2, be continuous along the inter-element boundaries. In other words, these interpolation functions must at least be C^1 continuous to satisfy inter-element continuity. Hence, it becomes difficult to find a conformal interpolation function satisfying C^1 continuity along the element interfaces. However, past investigations on elements using convergent but non-conformal shape functions are available. The key to a successful non-conformal shape function is known as the satisfaction of the "patch-test" or the "individual element test" as presented by Bergan and Nygard (1984). For example the "Free-Formulation" technique introduced by Bergan and Nygard and applied by Bergan and Wang (1984) for shear deformable elements may be a candidate for the finite element implementation of the present {3,2}-order plate theory.

5. Conclusions

In this analysis, a higher-order plate theory based on cubic expansion for in-plane displacements and a special form of quadratic expansion for the out-of-plane displacement component through the thickness is presented for the analysis of thick sandwich plates. Enforcing Reissner's definitions of weighted-average quantities, the number of variables describing the displacements at a point has been reduced from 11 to 7 (i.e., three translations, two out-of-plane rotations, and two higher-order transverse displacement modes representing the symmetric and anti-symmetric expansions in the transverse direction). The continuity of transverse stress fields has been achieved through a cubic polynomial that satisfies transverse stress equilibrium equations on the upper and lower surfaces of the panel. Comparison of the present theory with the exact solution shows close agreement for all stress components for typical material systems used in sandwich construction.

Appendix A

The traction forces and moments in Eq. (23) are defined as

$$(\hat{T}_{x0}, \hat{M}_{x0}, \hat{T}_{x1}, \hat{M}_{x1}) = \int_{-h}^{h} T_{x0}(P_0, P_1, P_2, P_3)\,\mathrm{d}z$$

$$(\hat{T}_{y0}, \hat{M}_{y0}, \hat{T}_{y1}, \hat{M}_{y1}) = \int_{-h}^{h} T_{y0}(P_0, P_1, P_2, P_3)\,\mathrm{d}z$$

$$(\hat{Q}_{nz0}, \hat{Q}_{nz1}, \hat{Q}_{nz2}) = \int_{-h}^{h} T_{z0}\left(1, \xi, \xi^2 - \frac{1}{5}\right)\mathrm{d}z$$

The vectors, $\mathbf{q}'_{\varepsilon}$, $\mathbf{q}''_{\varepsilon}$, $\mathbf{q}'_{\kappa}$, and $\mathbf{q}''_{\kappa}$ in Eq. (18) are obtained from the matrix operation

$$\begin{bmatrix} \mathbf{q}'_{\varepsilon} & \mathbf{q}'_{\kappa} \\ \mathbf{q}''_{\varepsilon} & \mathbf{q}''_{\kappa} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{t}'_{\varepsilon} & \mathbf{t}'_{\kappa} \\ \mathbf{t}''_{\varepsilon} & \mathbf{t}''_{\kappa} \end{bmatrix}$$

where s_{ij} $(i, j = 1, 2)$ and $\mathbf{t}'_{\alpha}$ and $\mathbf{t}''_{\alpha}$ $(\alpha = \varepsilon, \kappa)$ are given as

$$[s_{11}, s_{12}, s_{22}] = \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} [1, \phi, \phi^2] s_{33}^{(k)^2} \, \mathrm{d}z_k$$

and

$$\mathbf{t}'_\alpha = \{t'_{k\alpha}\} \quad \text{and} \quad \mathbf{t}''_\alpha = \{t''_{k\alpha}\}, \qquad (k = 1, 7; \alpha = \varepsilon, \kappa)$$

in which

$$\left.\begin{aligned} t'_{j\varepsilon} &= \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \varphi_{j\varepsilon} \hat{R}_{j3}^{(k)} s_{33}^{(k)} \, \mathrm{d}z_k, \quad & t'_{j\kappa} &= \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \varphi_{j\kappa} \hat{R}_{j3}^{(k)} s_{33}^{(k)} \, \mathrm{d}z_k \\ t''_{j\varepsilon} &= \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \varphi_{j\varepsilon} \phi \hat{R}_{j3}^{(k)} s_{33}^{(k)} \, \mathrm{d}z_k, \quad & t''_{j\kappa} &= \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \varphi_{j\kappa} \phi \hat{R}_{j3}^{(k)} s_{33}^{(k)} \, \mathrm{d}z_k \end{aligned}\right\} \quad (j = 1, 7; \text{no summation on } j)$$

with

$$\left\{\hat{R}_{j3}^{(k)}\right\} = \left\{R_{13}^{(k)}, R_{23}^{(k)}, 1, R_{63}^{(k)}, R_{13}^{(k)}, R_{23}^{(k)}, R_{63}^{(k)}\right\}$$

and

$$\{\varphi_{j\varepsilon}\} = \{P_0, P_0, 1, P_0, P_2, P_2, P_2\}, \qquad \{\varphi_{j\kappa}\} = \{P_1, P_1, 1, P_1, P_3, P_3, P_3\}$$

The components of matrices **A**, **B**, **D**, and **G** in Eq. (27) are obtained as

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \{\hat{C}_{ij}^{(k)} \varphi_{i\varepsilon} \varphi_{j\varepsilon} + s_{33}^{(k)} \psi_{i\varepsilon} \psi_{j\varepsilon}\} \, \mathrm{d}z_k \\ B_{ij} &= \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \{\hat{C}_{ij}^{(k)} \varphi_{i\varepsilon} \varphi_{j\kappa} + s_{33}^{(k)} \psi_{i\varepsilon} \psi_{j\kappa}\} \, \mathrm{d}z_k \\ D_{ij} &= \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \{\hat{C}_{ij}^{(k)} \varphi_{i\kappa} \varphi_{j\kappa} + s_{33}^{(k)} \psi_{i\kappa} \psi_{j\kappa}\} \, \mathrm{d}z_k \end{aligned}\right\} \quad (i, j = 1, 7; \text{no summation on } i \text{ and } j)$$

and

$$G_{ij} = \sum_{k=1}^{K} \int_{h_{k-1}}^{h_k} \left\{ \left[\frac{5}{4}(1 - \xi^2)\right]^2 \hat{C}_{i+3,j+3} \right\} \mathrm{d}z_k, \quad (i, j = 1, 2)$$

in which $\hat{C}_{ij}^{(k)}$ is defined in matrix form as

$$\left[\hat{C}_{ij}^{(k)}\right] = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & 0 & \bar{C}_{16} & \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ & \bar{C}_{22} & 0 & \bar{C}_{26} & \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{26} \\ & & 0 & 0 & 0 & 0 & 0 \\ & & & \bar{C}_{66} & \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{66} \\ \text{sym.} & & & & \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{16} \\ & & & & & \bar{C}_{22} & \bar{C}_{26} \\ & & & & & & \bar{C}_{66} \end{bmatrix}^{(k)}$$

and the functions $\varphi_{j\varepsilon}$ and $\varphi_{j\kappa}$ $(j = 1, 7)$ are defined in the form

$$\left.\begin{aligned} \psi_{j\varepsilon} &= q'_{j\varepsilon} + \phi q''_{j\varepsilon} \\ \psi_{j\kappa} &= q'_{j\kappa} + \phi q''_{j\kappa} \end{aligned}\right\} \quad (j = 1, 7)$$

Note that matrices **A**, **D**, and **G** are symmetric whereas matrix **B** is non-symmetric due to the presence of coupled product terms.

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